

Restrictions on Gauge Groups in Noncommutative Gauge Theory

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We show that the gauge groups $SU(N)$, $SO(N)$ and $Sp(N)$ cannot be realized on a flat noncommutative manifold, while it is possible for $U(N)$.

I. INTRODUCTION

The Yang-Mills theories naturally arise as low energy limits of the theory of open strings. One can obtain Yang-Mills theories with different gauge groups by studying different D-brane configurations (see e.g. [1]). For instance, if we place N D-branes on top of each other in the flat space, the corresponding open string theory gives rise to the Yang-Mills theory with the gauge group $U(N)$. One can also obtain gauge theories with other gauge groups such as $SO(N)$ and $Sp(N)$ by using the orientifold construction. In more detail, one combines the spatial reflection $\sigma \rightarrow \pi - \sigma$ on the string world-sheet with the target space reflection, $X^\mu \rightarrow -X^\mu$, $\mu = 1, \dots, k$ and $X^\mu \rightarrow X^\mu$, $\mu = (k+1), \dots, 10$. It is the goal of this note to study which gauge groups can be realized in the presence of the background B -field when the brane world-volume turns into a noncommutative space [2,3,4].

Interaction with the B -field introduces an extra term into the Polyakov action of the string [4],

$$\Delta S = \frac{-i}{2} \int_{\Sigma} B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu. \quad (1)$$

Here $a, b = 1, 2$ are world-sheet indices, Σ is the world-sheet and $B_{\mu\nu} = -B_{\nu\mu}$ is the B -field on the target space. One requires the expression (1) to be invariant with respect to the orientifold reflection. This implies the following transformation rules for components of the B -field,

$$\begin{aligned} B_{|||} &\rightarrow -B_{|||} & B_{\perp\perp} &\rightarrow -B_{\perp\perp} \\ B_{||\perp} &\rightarrow B_{||\perp} & B_{\perp||} &\rightarrow B_{\perp||}. \end{aligned} \quad (2)$$

Here the symbols $||$ and \perp stand for the target space indices $\mu = 1, \dots, k$ and $\mu = (k+1), \dots, 10$, respectively. In the orientifold construction we finally let the branes lie on the orientifold. The continuity of the B -field implies that the B -field on the brane, $B_{|||}$, vanishes. Hence, the brane world-volumes are commutative since it is $B_{|||}$ which is responsible for the noncommutativity [2,3,4]. This consideration indicates that one should encounter difficulties in the construction of the gauge theories with gauge groups $SO(N)$ and $Sp(N)$ on noncommutative spaces. Somewhat surprisingly, we find that the reduction from

$U(N)$ to $SU(N)$ in the framework of noncommutative geometry also fails.

II. THE CLOSURE OF CLASSICAL LIE ALGEBRAS UNDER THE MOYAL COMMUTATOR

In the flat case the presence of a constant B -field turns the D-branes into noncommutative spaces, with the ordinary pointwise multiplication of functions replaced by the Moyal product,

$$\begin{aligned} (X * Y)(x) &= \exp\left(\frac{i}{2} \theta^{ij} \partial_i^x \partial_j^y\right) X(x) Y(y)|_{x=y} = \\ &= XY + \frac{i}{2} \theta^{ij} \partial_i X \partial_j Y + \dots \end{aligned} \quad (3)$$

Here X and Y are functions on the D-brane world-volume, and θ^{ij} is a real-valued constant antisymmetric tensor constructed of the metric and B -field [3]. The Moyal product naturally extends to N by N matrices, formula (3) still applies. One can also introduce the Moyal commutator by the formula,

$$[X, Y]_* = X * Y - Y * X. \quad (4)$$

In what follows we check whether the matrix Lie algebras of the classical Lie groups $SO(N)$, $U(N)$, $SU(N)$ and $Sp(N)$ are closed under the Moyal commutator. We choose to work in the fundamental representation of these Lie algebras.

The Lie algebra of $U(N)$ consists of anti-Hermitian matrices, $\overline{X^t} = -X$, where the bar stands for complex conjugation. We first show that this algebra is closed under the Moyal commutator. The key observation is the following property of the Moyal product,

$$\overline{(X * Y)^t} = \overline{Y^t} * \overline{X^t}. \quad (5)$$

By using the ordinary rules for the transpose of matrices we get,

$$\begin{aligned} (X * Y)^t &= Y^t X^t + \\ &+ \frac{i}{2} \theta^{ij} \partial_j Y^t \partial_i X^t - \frac{1}{8} \theta^{*ij} \theta^{*kl} \partial_j \partial_l Y^t \partial_k \partial_i X^t + \dots \end{aligned} \quad (6)$$

The construction for higher order terms is obvious. Now we apply the complex conjugation and rename the indices of θ to obtain,

$$\overline{(X * Y)^t} = \overline{Y^t X^t} + \frac{i}{2} \theta^{ij} \partial_i \overline{Y^t} \partial_j \overline{X^t} - \quad (7)$$

$$\frac{1}{8} \theta^{ij} \theta^{kl} \partial_i \partial_k \overline{Y^t} \partial_j \partial_l \overline{X^t} + \dots = \overline{Y^t * X^t}. \quad (8)$$

Taking into account $\overline{X^t} = -X$ and $\overline{Y^t} = -Y$ yields,

$$\overline{[X, Y]_*^t} = \overline{(X * Y)^t} - \overline{Y * X^t} = \quad (9)$$

$$= \overline{Y^t * X^t} - \overline{X^t * Y^t} = \quad (10)$$

$$= Y * X - X * Y = -[X, Y]_* \quad (11)$$

which shows that the algebra $U(N)$ is closed under the Moyal commutator.

We now turn to the algebras of $SO(N)$, $SU(N)$ and $Sp(N)$. We first show that for $N = 2$ these algebras are not closed with respect to the Moyal commutator.

The counter examples for both $SO(2)$ and $Sp(2)$ are given by formulas,

$$X = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}. \quad (12)$$

and the counterexample for $SU(2)$ is

$$X = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} Y = \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix}. \quad (13)$$

Here α and β are coordinates on the manifold chosen so that $\theta^{\alpha\beta} \neq 0$. This can always be done unless $\theta = 0$

and the Moyal product coincides with the ordinary multiplication of matrix-valued functions. With X and Y as given above one can easily compute the Moyal commutator since all derivatives of order higher than one vanish. The result for both counter examples is

$$[X, Y]_* = i\theta^{\alpha\beta} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (14)$$

Note that this matrix has a nonvanishing trace. Since the Lie algebras of $SO(2)$, $SU(2)$ and $Sp(2)$ consist of traceless matrices, we conclude that they are not closed under the Moyal commutator. This also applies to $SO(N)$, $SU(N)$ and $Sp(N)$ for arbitrary N because they contain $SO(2)$, $SU(2)$ and $Sp(2)$ as subgroups.

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